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On the solutions of the dKP equation: the nonlinear Riemann Hilbert problem, longtime behaviour, implicit solutions and wave breaking

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Abstract

We have recently solved the inverse scattering problem for one-parameter families of vector fields, and used this result to construct the formal solution of the Cauchy problem for a class of integrable nonlinear partial differential equations in multidimensions, including the second heavenly equation of Plebanski and the dispersionless Kadomtsev–Petviashvili (dKP) equation. We showed, in particular, that the associated inverse problems can be expressed in terms of nonlinear Riemann–Hilbert problems on the real axis. In this paper, we make use of the nonlinear Riemann–Hilbert problem of dKP (i) to construct the longtime behaviour of the solutions of its Cauchy problem; (ii) to characterize a class of implicit solutions; (iii) to elucidate the spectral mechanism causing the gradient catastrophe of localized solutions of dKP, at finite time as well as in the longtime regime, and the corresponding universal behaviours near breaking.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It was observed long ago [1] that the commutation of multidimensional vector fields can generate integrable nonlinear partial differential equations (PDEs) in arbitrary dimensions. An important example of PDEs of this type is the dispersionless Kadomtsev–Petviashvili (dKP) equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \quad (1)$$

arising from the commutation

$$[\hat{L}_1, \hat{L}_2] = 0 \tag{2}$$

of the following pair of one-parameter families of vector fields:

$$\begin{aligned} \hat{L}_1 &\equiv \partial_y + \lambda \partial_x - u_x \partial_\lambda, \\ \hat{L}_2 &\equiv \partial_t + (\lambda^2 + u) \partial_x + (-\lambda u_x + u_y) \partial_\lambda, \end{aligned} \tag{3}$$

λ being the spectral parameter.

The dKP equation is the x -dispersionless limit of the celebrated Kadomtsev–Petviashvili (KP) equation [2] $(u_t + uu_x + u_{xxx})_x + \sigma u_{yy} = 0$, $\sigma = \pm 1$, and therefore it describes, for instance, the evolution of small amplitude, nearly one-dimensional waves in shallow water [3] near the shore, when the x -dispersion can be neglected. It is also a model equation in the description of unsteady motion in transonic flow [4] and in the nonlinear acoustics of confined beams [5].

We remark that, if the y -dispersion term u_{yy} is also negligible in (1), the dKP equation reduces to the celebrated (1+1)-dimensional Hopf equation: $u_t + uu_x = 0$, the integrable prototype model for the description of the gradient catastrophe (breaking) of a localized one-dimensional wave. Therefore a natural question is whether the dKP equation can be viewed as the integrable universal model in the description of the gradient catastrophe of a localized two-dimensional wave. As far as we know, no precise results are known in this direction, although some formal considerations have been made, in the framework of unsteady motions in transonic flow [6]. The following issues are, in particular, relevant. (i) Do localized initial data evolving according to dKP break? (ii) Do small initial data also break? (ii) If breaking occurs, does it take place at a point of the (x, y) -plane or on a line? (iii) Do the analytic and geometric aspects of the breaking exhibit universal features, like in the one-dimensional case? (iv) How are the breaking features connected with the dKP initial data? In this paper all these questions will find a proper answer.

Dispersionless (or quasi-classical) limits of integrable PDEs, having dKP as prototype example, arise in various problems of *Mathematical Physics* and are intensively studied in the recent literature (see, f.i., [7–21]). The Lax representation (3) and the Hamiltonian formulation for dKP can be found, for instance, in [9, 10]. An elegant integration scheme for dKP, applicable in general to nonlinear PDEs associated with Hamiltonian vector fields, was presented in [10]. A nonlinear $\bar{\partial}$ -dressing was developed in [15]. Special classes of nontrivial solutions were also derived (see, f.i., [14, 16]).

The inverse scattering transform for one-parameter families of multidimensional vector fields has been developed in [22] (see also [23]). This theory, introducing interesting novelties with respect to the classical inverse scattering transform for soliton equations [3, 24], has allowed one to construct the formal solution of the Cauchy problems for the heavenly equation [25] in [22] and for the following novel system of PDEs,

$$\begin{aligned} u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\ v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0, \end{aligned} \tag{4}$$

arising from the commutation of the vector fields

$$\begin{aligned} \tilde{L}_1 &\equiv \partial_y + (\lambda + v_x) \partial_x - u_x \partial_\lambda, \\ \tilde{L}_2 &\equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y) \partial_x + (-\lambda u_x + u_y) \partial_\lambda, \end{aligned} \tag{5}$$

in [26]. The Cauchy problem for the $v = 0$ reduction of (4), the dKP equation (1), was also presented in [26], while the Cauchy problem for the $u = 0$ reduction of (4), an integrable system introduced in [27], was given in [28]. Due to the ring property of the space of

eigenfunctions associated with vector fields, the inverse problem can be formulated in three different distinguished ways, one of them being expressed in terms of a linear integral equation.

In this paper we show that one of these formulations, the nonlinear Riemann–Hilbert (RH) problem on the real axis, is an efficient way (i) to construct the longtime behaviour of the solutions of the Cauchy problem for dKP; (ii) to characterize a class of implicit solutions of dKP, parametrized by an arbitrary spectral function of a single variable; (iii) to establish that localized solutions of dKP generically break (if small, they break in the longtime regime), and to study in great detail the universal features of the wave breaking: the similarity solution before breaking, the vertical inflection at breaking, and the development of a three-valued region in the (x, y) -plane delimited by a caustic after breaking; (iv) to connect the analytic and universal features of this gradient catastrophe with the initial data, via the spectral transform developed in [26].

The paper is organized as follows. In section 2, we present the nonlinear RH dressing of the system (4) and of its dKP reduction (1), showing that it provides an implicit spectral representation of the solution depending on the solution itself, leading to the breaking of localized solutions at finite t . In section 3, we make use of such a RH problem to construct the longtime behaviour of the solutions of the Cauchy problem for the dKP equation, showing that, also asymptotically, the above spectral mechanism for breaking is present. In section 4, we select a class of spectral data for which the vector RH problem decouples and linearizes, leading to a class of implicit solutions of dKP. In section 5, we study the analytic aspects of the longtime breaking of a localized initial condition, showing that this problem is connected to the wave breaking of a localized two-dimensional wave evolving according to the (1+1)-dimensional Hopf equation. In section 6, we describe the analytic aspects of the wave breaking at finite time.

2. The nonlinear RH inverse problem

From the inverse scattering transform developed in [26] for the system (4) and for its dKP reduction (1), one extracts the following inverse problem.

Proposition. *Consider the vector nonlinear RH problem on the real line:*

$$\vec{\pi}^+(\lambda) = \vec{\mathcal{R}}(\vec{\pi}^-(\lambda)), \quad \lambda \in \mathbb{R}, \quad (6)$$

where the solutions $\vec{\pi}^\pm(\lambda) = (\pi_1^\pm(\lambda), \pi_2^\pm(\lambda)) \in \mathbb{C}^2$ are two-dimensional vector functions analytic in the upper and lower halves of the complex λ plane, normalized in the following way:

$$\vec{\pi}^\pm(\lambda) = \begin{pmatrix} -\lambda^2 t - \lambda y + x - 2ut \\ \lambda \end{pmatrix} + \vec{O}(\lambda^{-1}), \quad |\lambda| \gg 1, \quad (7)$$

where

$$u = \lim_{\lambda \rightarrow \infty} (\lambda(\pi_2^\pm(\lambda) - \lambda)), \quad (8)$$

and $\vec{\mathcal{R}}(\vec{\zeta}) = (\mathcal{R}_1(\zeta_1, \zeta_2), \mathcal{R}_2(\zeta_1, \zeta_2)) \in \mathbb{C}^2$, $\vec{\zeta} \in \mathbb{C}^2$ are given differentiable spectral data.

Then, assuming that the above RH problem and its linearized form $\vec{\sigma}^+ = J\vec{\sigma}^-$ are uniquely solvable, where J is the Jacobian matrix of the transformation (6): $J_{ij} = \partial \mathcal{R}_i / \partial \zeta_j$, $i, j = 1, 2$, the solutions $\vec{\pi}^\pm(\lambda)$ of the RH problem (6) are common eigenfunctions of the vector fields (5): $\vec{L}_j \vec{\pi}^\pm = \vec{0}$, $j = 1, 2$, where

$$v(x, y, t) = -yu - \lim_{\lambda \rightarrow \infty} \lambda(\pi_1^\pm + t\pi_2^{\pm 2} - (x - \lambda y)), \quad (9)$$

and u, v are solutions of the nonlinear system (4).

If the spectral data $\vec{\mathcal{R}}(\vec{\zeta})$ satisfy the reality constraint

$$\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\vec{\zeta})}) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2, \quad (10)$$

then the solutions u, v are real: $u, v \in \mathbb{R}$. If, in addition, the transformation $\vec{\zeta} \rightarrow \vec{\mathcal{R}}(\vec{\zeta})$ is canonical:

$$\{\mathcal{R}_1, \mathcal{R}_2\}_{\vec{\zeta}} = 1, \quad (11)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to the variables (ζ_1, ζ_2) , then $v = 0$, $\vec{\pi}^\pm(\lambda)$ are common eigenfunctions of the vector fields (3): $\hat{L}_j \vec{\pi}^\pm = \vec{0}$, $j = 1, 2$ and u satisfies the dKP equation (1).

Proof. One easily shows that the vectors $\tilde{L}_j \vec{\pi}^\pm$, $j = 1, 2$, solve the linearized RH problem: $(\tilde{L}_j \vec{\pi}^+) = J(\tilde{L}_j \vec{\pi}^-)$, and that $\tilde{L}_j \vec{\pi}^\pm \rightarrow 0$ as $\lambda \rightarrow \infty$. By uniqueness, we infer that $\vec{\pi}^\pm$ are eigenfunctions of the operators \tilde{L}_j : $\tilde{L}_j \vec{\pi}^\pm = 0$, $j = 1, 2$, implying that u and v are solutions of the nonlinear system (4).

From the reality constraint (10) it follows that $\overline{\vec{\pi}^+} = \vec{\pi}^-$, implying $u, v \in \mathbb{R}$. From the dKP constraint (11) it follows that $\{\pi_1^+, \pi_2^+\} = \{\pi_1^-, \pi_2^-\}$, $\lambda \in \mathbb{R}$, where $\{\cdot, \cdot\}$ is now the Poisson bracket with respect to the variables (x, λ) . Since $\{\pi_1^\pm, \pi_2^\pm\} \rightarrow 1$, as $\lambda \rightarrow \infty$, the analyticity properties of the eigenfunctions imply that $\{\pi_1^+, \pi_2^+\} = \{\pi_1^-, \pi_2^-\} = 1$. At last, applying \tilde{L}_j , $j = 1, 2$ to this equation, one obtains the Hamiltonian constraint $v = 0$, and system (4) reduces to the dKP equation. \square

From the integral equations

$$\begin{aligned} \pi_1^-(\lambda) &= -\lambda^2 t - \lambda y + x - 2ut + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda - i0)} R_1(\pi_1^-(\lambda'), \pi_2^-(\lambda')), \\ \pi_2^-(\lambda) &= \lambda + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda - i0)} R_2(\pi_1^-(\lambda'), \pi_2^-(\lambda')), \end{aligned} \quad (12)$$

characterizing the solutions of the RH problem (6), and from definition (8), one obtains the following spectral characterization of the solution u :

$$u = F(x - 2ut, y, t) \in \mathbb{R}, \quad (13)$$

where the spectral function F , defined by

$$F(\xi, y, t) = - \int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2(\pi_1^-(\lambda; \xi, y, t), \pi_2^-(\lambda; \xi, y, t)), \quad (14)$$

$$\mathcal{R}_j(\zeta_1, \zeta_2) = \zeta_j + R_j(\zeta_1, \zeta_2), \quad j = 1, 2,$$

is connected with the initial data $u(x, y, 0)$ via the direct spectral transform developed in [26].

We remark that, while the inverse spectral transform of most of the known integrable PDEs provides a spectral representation of the solution involving, as parameters, the spacetime coordinates, the inverse problem of dKP provides a spectral representation (13), (14) of the solution involving, as parameter, also the solution u itself, in the combination $(x - 2ut)$. This is the spectral mechanism for the breaking of a generic localized initial condition at finite time t . We postpone to sections 5 and 6 a detailed analysis of the analyticity aspects of such a breaking.

We also remark that the mechanism responsible for this feature is that the vector field \hat{L}_2 is quadratic in λ and, at the same time, it contains the partial differential operator ∂_λ . Due to that, the normalization of the analytic eigenfunction of \hat{L}_2 involves the coefficient u of the vector field. It is easy to see that these properties are shared by the whole dKP hierarchy, associated with time operators \hat{L}_n involving higher powers of λ . An example of integrable system not associated with the commutation of vector fields, but exhibiting an inverse problem in which the solution is implicit, is the Harry Dym equation [29, 30], connected with the KdV equation by a hodograph transformation.

3. Asymptotics of the dKP equation

In this section, we concentrate on the study of the longtime behaviour of the solutions of dKP. Of course, this study is meaningful only if no breaking takes place before, at finite time. If the initial condition is, for instance, small, the nonlinear term in (1) becomes important only in the longtime regime, and no breaking takes place before.

As we shall see in the following, the spectral mechanism causing the breaking of a localized initial condition evolving according to the dKP equation is present also in the longtime regime.

We study the longtime $t \gg 1$ regime in the space regions

$$x = \tilde{x} + v_1 t, \quad y = v_2 t, \quad \tilde{x} - 2ut, \quad v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t \gg 1. \quad (15)$$

Then the system of nonlinear integral equations (12) is conveniently rewritten in terms of the functions $\phi_j, j = 1, 2$, defined by

$$\phi_1(\lambda) = \pi_1^-(\lambda) + \lambda^2 t + \lambda y - (x - 2ut), \quad \phi_2(\lambda) = \pi_2^-(\lambda) - \lambda, \quad (16)$$

as follows,

$$\phi_j(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda - i0)} R_j(\tilde{x} - 2ut - (\lambda' - \lambda_+)(\lambda' - \lambda_-)t + \phi_1(\lambda'), \lambda' + \phi_2(\lambda')), \quad j = 1, 2, \quad (17)$$

where

$$\lambda_{\pm} = -\frac{v_2}{2} \pm \sqrt{v_1 + \frac{v_2^2}{4}}. \quad (18)$$

The fast decay, in t , of $\phi_j, j = 1, 2$, due to the linear growth in t of the first argument of $R_j, j = 1, 2$, is partially contrasted if $\lambda_+ = \lambda_-$; i.e., on the parabola:

$$x + \frac{y^2}{4t} = \tilde{x} \quad \left(v_1 + \frac{v_2^2}{4} = 0 \right) \quad (19)$$

of the (x, y) -plane.

On such a parabola the integral equations read

$$\phi_j(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda - i0)} \times R_j\left(\tilde{x} - 2ut - \left(\lambda' + \frac{v_2}{2}\right)^2 t + \phi_1(\lambda'), \lambda' + \phi_2(\lambda')\right), \quad j = 1, 2. \quad (20)$$

Since, in this case, the main contribution to the integrals occurs when $\lambda' \sim -v_2/2$, we make the change of variable $\lambda' = -v_2/2 + \mu'/\sqrt{t}$, obtaining:

$$\phi_j(\lambda) = \frac{1}{2\pi i \sqrt{t}} \int_{\mathbb{R}} \frac{d\mu'}{\mu'/\sqrt{t} - (\lambda + v_2/2 - i0)} R_j\left(\tilde{x} - 2ut - \mu'^2 + \phi_1\left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}}\right), -\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} + \phi_2\left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}}\right)\right), \quad j = 1, 2. \quad (21)$$

If $|\lambda + v_2/2| \gg t^{-1/2}$, equation (21) implies that $\phi_j(\lambda) = O(t^{-1/2}), j = 1, 2$:

$$\phi_j(\lambda) \sim -\frac{1}{2\pi i \sqrt{t}(\lambda + v_2/2 - i0)} \int_{\mathbb{R}} d\mu' R_j\left(\tilde{x} - 2ut - \mu'^2 + \phi_1\left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}}\right), -\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} + \phi_2\left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}}\right)\right), \quad j = 1, 2, \quad (22)$$

and, via (13) and (14), that

$$u = -\frac{1}{2\pi i \sqrt{t}} \int_{\mathbb{R}} d\mu R_2 \left(\tilde{x} - 2ut - \mu^2 + \phi_1 \left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} \right), -\frac{v_2}{2} + \phi_2 \left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} \right) \right) + o \left(\frac{1}{\sqrt{t}} \right). \quad (23)$$

For $\lambda + v_2/2 = \mu/\sqrt{t}$, $\mu = O(1)$, instead, $\phi_j(\lambda)$, $j = 1, 2$ are $O(1)$:

$$\phi_j \left(-\frac{v_2}{2} + \frac{\mu}{\sqrt{t}} \right) \sim \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu'}{\mu' - (\mu - i0)} R_j \left(\tilde{x} - 2ut - \mu'^2 + \phi_1 \left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} \right), -\frac{v_2}{2} + \phi_2 \left(-\frac{v_2}{2} + \frac{\mu'}{\sqrt{t}} \right) \right), \quad j = 1, 2. \quad (24)$$

Therefore it is not possible to neglect, in the integral equations (21), $\phi_{1,2}$ in the arguments of $R_{1,2}$; it follows that these integral equations remain nonlinear also in the longtime regime.

The above results can be summarized as follows.

On the parabola (19), in the spacetime regions (15), the longtime behaviour of the solution of the dKP equation is given by

$$u = \frac{1}{\sqrt{t}} G \left(x + \frac{y^2}{4t} - 2ut, \frac{y}{2t} \right) + o \left(\frac{1}{\sqrt{t}} \right), \quad (25)$$

where

$$G(\xi, \eta) = -\frac{1}{2\pi i} \int_{\mathbb{R}} d\mu R_2(\xi - \mu^2 + a_1(\mu; \xi, \eta), -\eta + a_2(\mu; \xi, \eta)), \quad (26)$$

R_2 is the second component of the spectral data \vec{R} , and $a_{1,2}(\mu; \xi, \eta)$ are the unique solutions of the nonlinear integral equations

$$a_j(\mu; \xi, \eta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\mu'}{\mu' - (\mu - i0)} R_j(\xi - \mu'^2 + a_1(\mu'; \xi, \eta), -\eta + a_2(\mu'; \xi, \eta)), \quad j = 1, 2. \quad (27)$$

We remark that the system of integral equations (27) characterizes the solutions of the following vector nonlinear RH problem on the real μ -axis:

$$\begin{aligned} \vec{A}^+(\mu; \xi, \eta) &= \vec{A}^-(\mu; \xi, \eta) + \vec{R}(\vec{A}^-(\mu; \xi, \eta)), & \mu \in \mathbb{R}, \\ \vec{A}^\pm(\mu; \xi, \eta) &= \begin{pmatrix} \xi - \mu^2 \\ -\eta \end{pmatrix} + \vec{O}(\mu^{-1}), & |\mu| \gg 1, \end{aligned} \quad (28)$$

with

$$\vec{a}(\mu; \xi, \eta) = \vec{A}^-(\mu; \xi, \eta) - \begin{pmatrix} \xi - \mu^2 \\ -\eta \end{pmatrix}. \quad (29)$$

Outside the parabola, the solution decays faster than $1/\sqrt{t}$.

We remark that equation (25) has been obtained under the hypothesis that both arguments of G are $O(1)$; then $u = O(1/\sqrt{t})$. It follows that $2ut = O(\sqrt{t})$ and, consequently, also $\tilde{x} = x + y^2/(4t) = O(\sqrt{t})$, to balance the term $2ut$.

From equation (25) it follows that the spectral mechanism causing the breaking of a localized initial condition evolving according to the dKP equation is present also in the longtime regime. The analytic aspects of the longtime breaking of dKP solutions are illustrated in section 5.

4. Particular solutions of the RH problem

In this section we construct a class of explicit solutions of the vector nonlinear RH problem (6) and, correspondingly, a class of implicit solutions of the dKP equation parametrized by an arbitrary real spectral function of one variable (the general solution would depend on an arbitrary real spectral function of two variables).

Suppose that the components of the spectral data $\tilde{\mathcal{R}}$ in (6) are given by:

$$\mathcal{R}_j(\zeta_1, \zeta_2) = \zeta_j e^{i(-)^{j+1} f(\zeta_1 \zeta_2)} - 1, \quad j = 1, 2, \quad (30)$$

in terms of the single real spectral function f of a single argument, depending on ζ_1 and ζ_2 only through their product.

Then the RH problem becomes

$$\begin{aligned} \pi_1^+ &= \pi_1^- e^{if(\pi_1^- \pi_2^-)}, & \lambda \in \mathbb{R}, \\ \pi_2^+ &= \pi_2^- e^{-if(\pi_1^- \pi_2^-)}, \end{aligned} \quad (31)$$

and the following properties hold.

- (i) The reality and the dKP constraints (10) and (11) are satisfied.
- (ii) $\pi_1^+ \pi_2^+ = \pi_1^- \pi_2^-$. Consequently, $(\pi_1^+ \pi_2^+)$ is just a polynomial in λ :

$$\pi_1^+ \pi_2^+ = \pi_1^- \pi_2^- = -t\lambda^3 - y\lambda^2 + (x - 3ut)\lambda - 2yu + 3t\partial_x^{-1}u_y \equiv w(\lambda), \quad (32)$$

and the vector nonlinear RH problem (6) decouples into two scalar, linear RH problems:

$$\pi_1^+ = \pi_1^- e^{if(w(\lambda))}, \quad \pi_2^+ = \pi_2^- e^{-if(w(\lambda))}. \quad (33)$$

- (iii) Since, from (33),

$$\pi_j^+ e^{i(-)^j f^+(\lambda)} = \pi_j^- e^{i(-)^j f^-(\lambda)}, \quad j = 1, 2, \quad (34)$$

where

$$f^\pm(\lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda \pm i0)} f(w(\lambda')), \quad (35)$$

also $\pi_j^\pm e^{i(-)^j f^\pm(\lambda)}$, $j = 1, 2$ are polynomials in λ . We expand them in powers of λ and introduce the notation

$$\langle \lambda^n f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^n f(w(\lambda)) d\lambda, \quad n \in \mathbb{N}. \quad (36)$$

From the positive power expansion it follows that

$$\begin{aligned} \pi_1^+ e^{-if^+(w(\lambda))} &= \pi_1^- e^{-if^-(w(\lambda))} \\ &= -t\lambda^2 - (y + t\langle f \rangle)\lambda + x - 2ut - y\langle f \rangle - t \left(\langle \lambda f \rangle + \frac{\langle f \rangle^2}{2} \right), \end{aligned} \quad (37)$$

$$\pi_2^+ e^{if^+(w(\lambda))} = \pi_2^- e^{if^-(w(\lambda))} = \lambda - \langle f \rangle,$$

implying the following explicit solution of the Riemann problem:

$$\begin{aligned} \pi_1^\pm &= \left[-\lambda^2 t - \lambda(y + t\langle f \rangle) + x - 2ut - y\langle f \rangle - t \left(\langle \lambda f \rangle + \frac{\langle f \rangle^2}{2} \right) \right] e^{if^\pm(w(\lambda))}, \\ \pi_2^\pm &= (\lambda - \langle f \rangle) e^{-if^\pm(w(\lambda))}. \end{aligned} \quad (38)$$

From the $1/\lambda$ terms of $\pi_j^+ e^{i(-)^j f^+(\lambda)}$, $j = 1, 2$, that must be zero, one finally obtains the two conditions

$$u = \langle \lambda f \rangle - \frac{\langle f \rangle^2}{2},$$

$$2t \partial_x^{-1} u_y = yu - (x - 2ut) \langle f \rangle + y \left(\langle \lambda f \rangle + \frac{\langle f \rangle^2}{2} \right) + t \left(\langle \lambda^2 f \rangle + \langle f \rangle \langle \lambda f \rangle + \frac{\langle f \rangle^3}{6} \right). \quad (39)$$

Since, through (32) and (36), $\langle \lambda^n f \rangle$, $n = 0, 1, 2$, depend implicitly on the solution u and on $(t \partial_x^{-1} u_y)$, the system (39) characterizes a class of implicit solutions of the dKP equation, parametrized by the arbitrary real spectral function f of a single variable.

5. Longtime breaking of dKP solutions

In this section we show that longtime solutions of dKP break. Of course this analysis is meaningful only when breaking does not take place before, at finite time. If, for instance, the localized initial condition is small, the nonlinear term of the dKP equation becomes relevant only in the longtime regime, and breaking can occur only when t is large.

Let $U(x, y, t)$ be the exact solution of the functional equation (25) at the leading order, i.e.:

$$U(x, y, t) = \frac{1}{\sqrt{t}} G \left(x + \frac{y^2}{4t} - 2Ut, \frac{y}{2t} \right), \quad (40)$$

where G , connected with the initial condition of dKP via the direct spectral problem introduced in [26], is a largely arbitrary differentiable function of two arguments. It is easy to verify that U is the general solution of the first order PDE in 2 + 1 dimensions:

$$U_t + \frac{y}{t} U_y - \frac{y^2}{4t^2} U_x + \frac{U}{2t} + U U_x = 0. \quad (41)$$

Equation (40) suggests introducing the convenient variables:

$$V = \sqrt{t} U,$$

$$\tilde{x} = x + \frac{y^2}{4t}, \quad \tilde{y} = \frac{y}{2t}, \quad \tilde{t} = 2\sqrt{t}, \quad (42)$$

transforming the PDE (41) into the (1+1)-dimensional Hopf equation:

$$V_{\tilde{t}} + V V_{\tilde{x}} = 0. \quad (43)$$

Then the longtime behaviour of the dKP solutions is reduced to the study of the evolution of a two-dimensional localized wave under the (1+1)-dimensional Hopf equation (43); its solution, defined implicitly by the equations

$$V = G(\xi, \tilde{y}), \quad (44a)$$

$$\xi = \tilde{x} - G(\xi, \tilde{y}) \tilde{t}, \quad (44b)$$

depends on \tilde{y} only parametrically.

Each small portion of the two-dimensional localized wave, characterized by its own amplitude V , travels with constant speed V in the \tilde{x} -direction. Equations (44b) define a two-parameter family of characteristic curves in the $(\tilde{x}, \tilde{y}, \tilde{t})$ -space, the parameters being ξ, \tilde{y} . Due to the localization properties of G , on each plane $\tilde{y} = \text{const}$, the characteristic curves obtained varying ξ intersect, giving rise to the breaking phenomenon. The first breaking will take place therefore at a specific point $(\tilde{x}_b, \tilde{y}_b, \tilde{t}_b)$ and, going back to the physical variables inverting the transformation (42), at a specific point (x_b, y_b) of the (x, y) -plane, at the breaking time t_b (see figure 1).

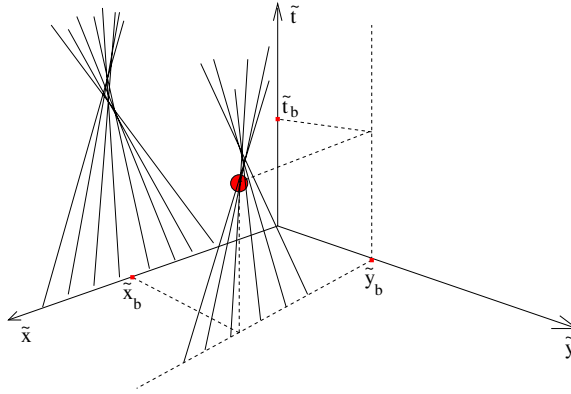


Figure 1. The two-parameter family of characteristic curves describing the evolution of 2D-waves according to the (1+1)-dimensional Hopf equation.

5.1. 2D—breaking according to the Hopf equation

In this section we discuss in detail the analytic aspects of the wave breaking of a localized two-dimensional wave evolving according to the Hopf equation (43), whose solution is defined implicitly by equations (44).

One solves (44b) with respect to the parameter ξ , obtaining $\xi(\tilde{x}, \tilde{y}, \tilde{t})$, and replaces it into (44a), to obtain the solution $V = G(\xi(\tilde{x}, \tilde{y}, \tilde{t}), \tilde{y})$. The inversion of equation (44b) is possible iff its ξ -derivative is different from zero. Therefore the singularity manifold (SM) of the two dimensional Cauchy problem for the Hopf equation is the two-dimensional manifold characterized by the equation

$$S(\xi, \tilde{y}, t) \equiv 1 + G_\xi(\xi, \tilde{y})\tilde{t} = 0 \Rightarrow \tilde{t} = -\frac{1}{G_\xi(\xi, \tilde{y})}. \tag{45}$$

Since

$$\nabla_{(\tilde{x}, \tilde{y})} V = \frac{\nabla_{(\xi, \tilde{y})} G(\xi, \tilde{y})}{1 + G_\xi(\xi, \tilde{y})\tilde{t}}, \tag{46}$$

the slopes of the localized wave become infinity (the so-called gradient catastrophe) on the SM, and the two-dimensional wave ‘breaks’.

Then the first breaking time \tilde{t}_b , and the corresponding characteristic parameters $\vec{\xi}_b = (\xi_b, \tilde{y}_b)$ are defined by

$$\tilde{t}_b = -\frac{1}{G_\xi(\vec{\xi}_b)} = \text{global min} \left(-\frac{1}{G_\xi(\xi, \tilde{y})} \right) > 0, \tag{47}$$

and characterized by the equations:

$$\begin{aligned} G_\xi(\vec{\xi}_b) < 0, & \quad G_{\xi\xi}(\vec{\xi}_b) = G_{\xi\tilde{y}}(\vec{\xi}_b) = 0, \\ G_{\xi\xi\xi}(\vec{\xi}_b) > 0, & \quad \alpha \equiv G_{\xi\xi\xi}(\vec{\xi}_b)G_{\xi\tilde{y}\tilde{y}}(\vec{\xi}_b) - G_{\xi\xi\tilde{y}}^2(\vec{\xi}_b) > 0. \end{aligned} \tag{48}$$

The corresponding point $\vec{\tilde{x}}_b = (\tilde{x}_b, \tilde{y}_b)$ in which the first wave breaking takes place is, from (44b):

$$\tilde{x}_b = \xi_b + G(\vec{\xi}_b)\tilde{t}_b. \tag{49}$$

Now we evaluate equations (44b) and (45) near breaking, in the regime:

$$\tilde{x} = \tilde{x}_b + \tilde{x}', \quad \tilde{y} = \tilde{y}_b + \tilde{y}', \quad \tilde{t} = \tilde{t}_b + \tilde{t}', \quad \xi = \xi_b + \xi', \quad (50)$$

where $\tilde{x}', \tilde{y}', \tilde{t}', \xi'$ are small. Using (47)–(49), equation (44b) becomes, at the leading order, the following cubic equation in ξ' :

$$\xi'^3 + a(\tilde{y}')\xi'^2 + b(\tilde{y}', \tilde{t}')\xi' - \gamma X(\tilde{x}', \tilde{y}', \tilde{t}') = 0, \quad (51)$$

where

$$\begin{aligned} a(\tilde{y}') &= \frac{3G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}', & b(\tilde{y}', \tilde{t}') &= \frac{3}{G_{\xi\xi\xi}}[G_{\xi}\epsilon + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2], \\ X(\tilde{x}', \tilde{y}', \tilde{t}') &= \tilde{x}' - G(\xi_b, \tilde{y}_b + \tilde{y}')\tilde{t}' - [G(\xi_b, \tilde{y}_b + \tilde{y}') - G]\tilde{t}_b \\ &\sim \tilde{x}' + \frac{G_{\tilde{y}}}{G_{\xi}}\tilde{y}' - G\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}}}{2G_{\xi}}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_{\xi}}\tilde{y}'^3, & \gamma &= \frac{6|G_{\xi}|}{G_{\xi\xi\xi}}, \end{aligned} \quad (52)$$

with

$$\epsilon \equiv \frac{2\tilde{t}'}{\tilde{t}_b}, \quad |\epsilon| \ll 1, \quad (53)$$

corresponding to the maximal balance

$$|\xi'|, |\tilde{y}'| = O(|\epsilon|^{1/2}), \quad |X| = O(|\epsilon|^{3/2}). \quad (54)$$

In (52) and in the rest of this section, all partial derivatives of G whose arguments are not indicated are meant to be evaluated at $\tilde{\xi}_b = (\xi_b, \tilde{y}_b)$.

The three roots of the cubic are given by the well-known Cardano formula:

$$\begin{aligned} \xi'_{0}(\tilde{x}', \tilde{y}', \tilde{t}') &= -\frac{a}{3} + (A_{+})^{\frac{1}{3}} + (A_{-})^{\frac{1}{3}}, \\ \xi'_{\pm}(\tilde{x}', \tilde{y}', \tilde{t}') &= -\frac{a}{3} - \frac{1}{2}((A_{+})^{\frac{1}{3}} + (A_{-})^{\frac{1}{3}}) \pm \frac{\sqrt{3}}{2}i((A_{+})^{\frac{1}{3}} - (A_{-})^{\frac{1}{3}}), \end{aligned} \quad (55)$$

where

$$A_{\pm} = R \pm \sqrt{\Delta} \quad (56)$$

and the discriminant Δ reads

$$\Delta = R^2 + Q^3, \quad (57)$$

with

$$\begin{aligned} Q(\tilde{y}', \tilde{t}') &= \frac{3b - a^2}{9} = -\frac{|G_{\xi}|}{G_{\xi\xi\xi}}\epsilon + \frac{\alpha}{G_{\xi\xi\xi}^2}\tilde{y}'^2, \\ R(\tilde{x}', \tilde{y}', \tilde{t}') &= \frac{\gamma}{2}X(\tilde{x}', \tilde{y}', \tilde{t}') + \frac{ab}{18} + \frac{a}{3}Q(\tilde{y}', \tilde{t}'). \end{aligned} \quad (58)$$

At the same order, function S in (45) becomes

$$S(\xi, \tilde{y}, t) = G_{\xi}\tilde{t}' + \frac{1}{2}[G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2]\tilde{t}_b. \quad (59)$$

Known ξ' as a function of $(\tilde{x}, \tilde{y}, \tilde{t})$ from the cubic (51), the solution V of the Hopf equation and its gradient are then approximated, near breaking, by the formulae:

$$\begin{aligned} V(\tilde{x}, \tilde{y}, \tilde{t}) &\sim G(\xi_b + \xi', \tilde{y}_b + \tilde{y}'), \\ \nabla_{(\tilde{x}, \tilde{y})} V &\sim \frac{\nabla_{(\xi', \tilde{y}')} G(\xi_b + \xi', \tilde{y}_b + \tilde{y}')}{G_{\xi}\tilde{t}' + \frac{1}{2}[G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2]\tilde{t}_b}. \end{aligned} \quad (60)$$

Together with the breaking point \tilde{x}_b , another distinguished point is the inflection point \tilde{x}_f :

$$\tilde{x}_f = (\tilde{x}_f(\tilde{t}'), \tilde{y}_b), \quad \tilde{x}_f(\tilde{t}') = \tilde{x}_b + G\tilde{t}' \tag{61}$$

at which $R = X = \tilde{y}' = a = \xi' = \xi'_{\tilde{x}\tilde{x}} \equiv \xi'_{\tilde{x}\tilde{y}} = 0$ and, consequently:

$$V = G, \quad \nabla_{(\tilde{x}, \tilde{y})} V = \frac{1}{\tilde{t}'} \left(1, \frac{G_{\tilde{y}}}{G_{\xi}} \right), \quad V_{\tilde{x}\tilde{x}} = V_{\tilde{x}\tilde{y}} = 0. \tag{62}$$

We observe that $\tilde{x}_f = \tilde{x}_b$ at $\tilde{t} = \tilde{t}_b$.

5.1.1. Before breaking. If $\tilde{t} < \tilde{t}_b$ ($\tilde{t}' < 0$), the coefficient Q , defined in (58), is strictly positive, due to (48); then the discriminant $\Delta = R^2 + Q^3$ is also strictly positive and only the root ξ'_0 is real. Correspondingly, the real solution of (43) is single valued and described by Cardano's formula. In addition, function \mathcal{S} in (59) is also strictly positive and $\nabla_{(\tilde{x}, \tilde{y})} V$ is finite $\forall \tilde{x}, \tilde{y}$.

To have a more explicit solution, we restrict the asymptotic region around the inflection point to the narrow strip :

$$|\tilde{y}'| = O(|\epsilon|^q), \quad |X(\tilde{x}', \tilde{y}', \tilde{t}')| = O(|\epsilon|^{p+1}), \tag{63}$$

where

$$\max\left(\frac{p+1}{2}, p\right) < q < p+1, \quad p > \frac{1}{2}, \tag{64}$$

obtaining

$$\begin{aligned} X &\sim \tilde{x}' + (G_{\tilde{y}}/G_{\xi})\tilde{y}' - G\tilde{t}' = \tilde{k} \cdot (\tilde{x} - \tilde{x}_f(\tilde{t}')) = O(|\epsilon|^{p+1}), \\ \xi' &\sim \frac{\tilde{x}' + (G_{\tilde{y}}/G_{\xi})\tilde{y}' - G\tilde{t}'}{G_{\xi}\tilde{t}'} = O(|\epsilon|^p), \end{aligned} \tag{65}$$

where

$$\tilde{k} = \left(1, \frac{G_{\tilde{y}}}{G_{\xi}} \right) \tag{66}$$

is normal to the strip and defines the breaking direction.

Replacing (65) into (60) we obtain that, in the narrow strip (63) and (64), the solution exhibits a universal behaviour, coinciding with the following exact similarity solution of the Hopf equation (43),

$$V \sim \frac{\tilde{x} - \tilde{x}_b + (G_{\tilde{y}}/G_{\xi})(\tilde{y} - \tilde{y}_b)}{\tilde{t} - \tilde{t}_b}, \tag{67}$$

and describes the plane tangent to the wave at the inflection point (see figure 2). In addition, the gradient of V behaves like

$$\nabla_{(\tilde{x}, \tilde{y})} V \sim \frac{\nabla_{(\xi, \tilde{y})} G}{G_{\xi}\tilde{t}' + \frac{G_{\xi\xi\xi}}{2|G_{\xi}|} \left(\frac{\tilde{x}' + (G_{\tilde{y}}/G_{\xi})\tilde{y}' - G\tilde{t}'}{\tilde{t}'} + \frac{G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}' \right)^2 + \frac{\alpha}{2|G_{\xi}|G_{\xi\xi\xi}}\tilde{y}'^2}, \tag{68}$$

implying that, in the narrow strip (63) and (64),

$$\nabla_{(\tilde{x}, \tilde{y})} V \sim \frac{1}{\tilde{t}'} \left(1, \frac{G_{\tilde{y}}}{G_{\xi}} \right) \tag{69}$$

while outside, in the region $|X| = O(|\epsilon|)$, it becomes finite. We finally remark that the tangent plane described by (67) passes through the breaking point.

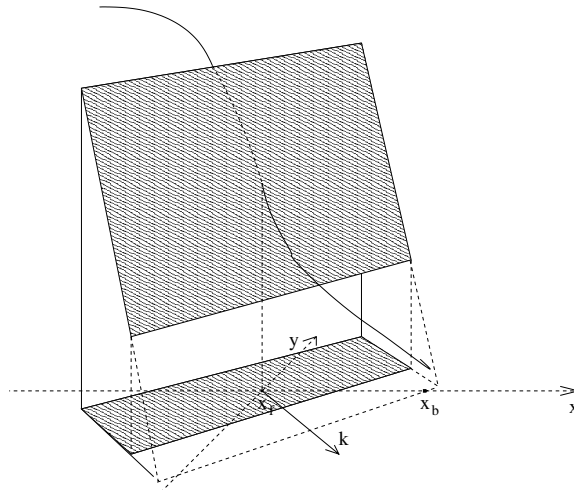


Figure 2. In a narrow strip of the plane, the similarity solution describes the plane tangent to the wave, immediately before breaking.

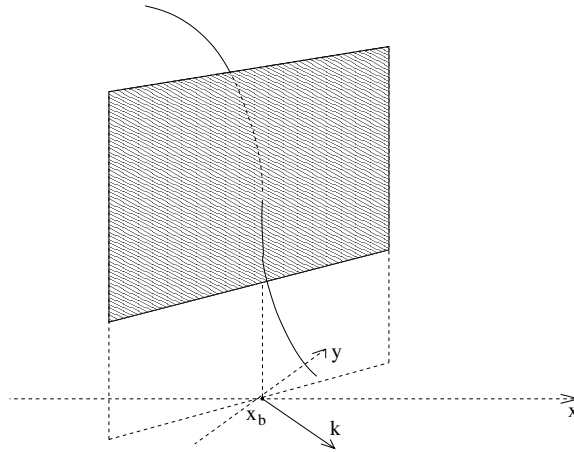


Figure 3. The vertical plane tangent to the wave at the breaking point.

5.1.2. *At breaking.* As $\tilde{t} \uparrow \tilde{t}_b$, the inflection point becomes the breaking point: $\tilde{x}_f \rightarrow \tilde{x}_b$, the above tangent plane (now tangent at the breaking point) becomes vertical, with equation $\tilde{k} \cdot (\tilde{x} - \tilde{x}_b) = 0$, the above strip reduces to the breaking point \tilde{x}_b , and the slopes $V_{\tilde{x}}, V_{\tilde{y}} \rightarrow \infty$ as $(\tilde{t} - \tilde{t}_b)^{-1}$ (see figure 3).

At the breaking time $\tilde{t} = \tilde{t}_b$, one can give an explicit description of the vertical inflection in the following two subcases:

- (i) $\tilde{y}' = 0$. In this case, the cubic (51) simplifies to $\xi'^3 = \gamma \tilde{x}'$, and the solution V exhibits the typical vertical inflection preceding the wave breaking:

$$V \sim G(\xi_b + \sqrt[3]{\gamma(\tilde{x} - \tilde{x}_b)}, \tilde{y}_b) \Rightarrow V_{\tilde{x}} \sim \frac{1}{3} \sqrt[3]{\frac{6|G_\xi|}{G_{\xi\xi\xi}}} \frac{G_\xi}{\sqrt[3]{(\tilde{x} - \tilde{x}_b)^2}}. \quad (70)$$

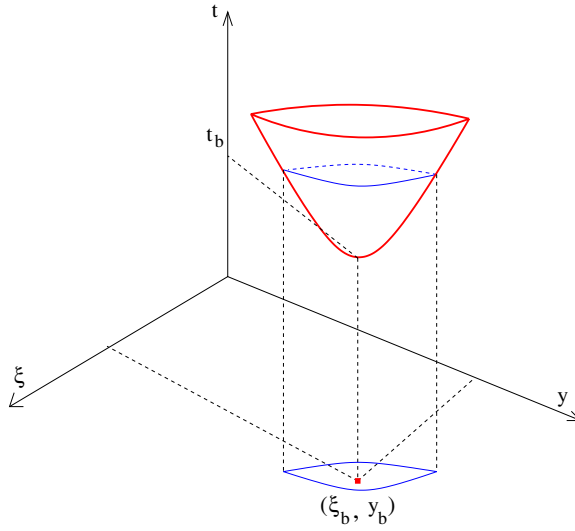


Figure 4. The two-dimensional singularity manifold $S(\xi, \eta, t) = 0$ near breaking.

(ii) $\tilde{x}' = 0$. In this case the cubic simplifies to $\xi'^3 - \gamma(G_{\tilde{y}}/G_{\xi})\tilde{y}' \sim 0$, and V describes a vertical inflection also w.r.t. \tilde{y} :

$$V \sim G \left(\xi_b + \sqrt[3]{\gamma \frac{G_{\tilde{y}}}{G_{\xi}} (\tilde{y} - \tilde{y}_b)}, \tilde{y}_b + \tilde{y}' \right) \Rightarrow V_{\tilde{y}} \sim \frac{1}{3} \sqrt[3]{\frac{6G_{\tilde{y}}}{G_{\xi\xi\xi}}} \frac{G_{\xi}}{\sqrt[3]{(\tilde{y} - \tilde{y}_b)^2}}. \quad (71)$$

5.1.3. *After breaking.* After breaking, the solution becomes three-valued in a domain of the (\tilde{x}, \tilde{y}) -plane, and does not describe any physics; nevertheless a detailed study of the multivalued region is important, in view of a proper regularization of the model, and/or in view of the introduction of a proper single-valued shock replacing the multivalued solution.

If $\tilde{t} > \tilde{t}_b$ ($\tilde{t}' > 0$), in regime (54), the SM equation $S = 0$:

$$G_{\xi\xi\xi}\xi'^2 + 2G_{\xi\xi\tilde{y}}\xi'\tilde{y}' + G_{\xi\tilde{y}\tilde{y}}\tilde{y}'^2 = |G_{\xi}| \epsilon \quad (72)$$

describes an elliptic paraboloid in the $(\xi, \tilde{y}, \tilde{t})$ space, with minimum at the point $(\tilde{\xi}_b, \tilde{t}_b)$ (see figure 4). The intersection of the elliptic paraboloid with any \tilde{t} -constant plane, $\tilde{t} > \tilde{t}_b$, defines an ellipse in the (ξ, \tilde{y}) -plane (see figure 4), whose coordinates vary in the following intervals of order $O(|\epsilon|^{1/2})$:

$$|\tilde{y}'| \leq \sqrt{\frac{|G_{\xi}|G_{\xi\xi\xi}\epsilon}{\alpha}}, \quad |\xi'| \leq \sqrt{\frac{|G_{\xi}|G_{\xi\tilde{y}\tilde{y}}\epsilon}{\alpha}}. \quad (73)$$

Eliminating ξ' from equations (72) and (51), one obtains the SM equation in spacetime coordinates:

$$\begin{aligned} & \left[3|G_{\xi}|G_{\xi\xi\xi}^2 \left(\tilde{x}' + \frac{G_{\tilde{y}}}{G_{\xi}}\tilde{y}' - G\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}}}{2G_{\xi}}\tilde{y}'^2 - G_{\tilde{y}}\tilde{y}'\tilde{t}' + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_{\xi}}\tilde{y}'^3 \right) \right. \\ & \quad \left. + \frac{G_{\xi\xi\xi}G_{\xi\xi\tilde{y}}G_{\xi\tilde{y}\tilde{y}}}{2} \left(\frac{G_{\xi}}{G_{\xi\tilde{y}\tilde{y}}}\epsilon + \tilde{y}'^2 \right) \tilde{y}' - \alpha G_{\xi\xi\tilde{y}} \left(\frac{|G_{\xi}|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2 \right) \tilde{y}' \right]^2 \\ & = \alpha^3 \left(\frac{|G_{\xi}|G_{\xi\xi\xi}}{\alpha}\epsilon - \tilde{y}'^2 \right)^3, \end{aligned} \quad (74)$$

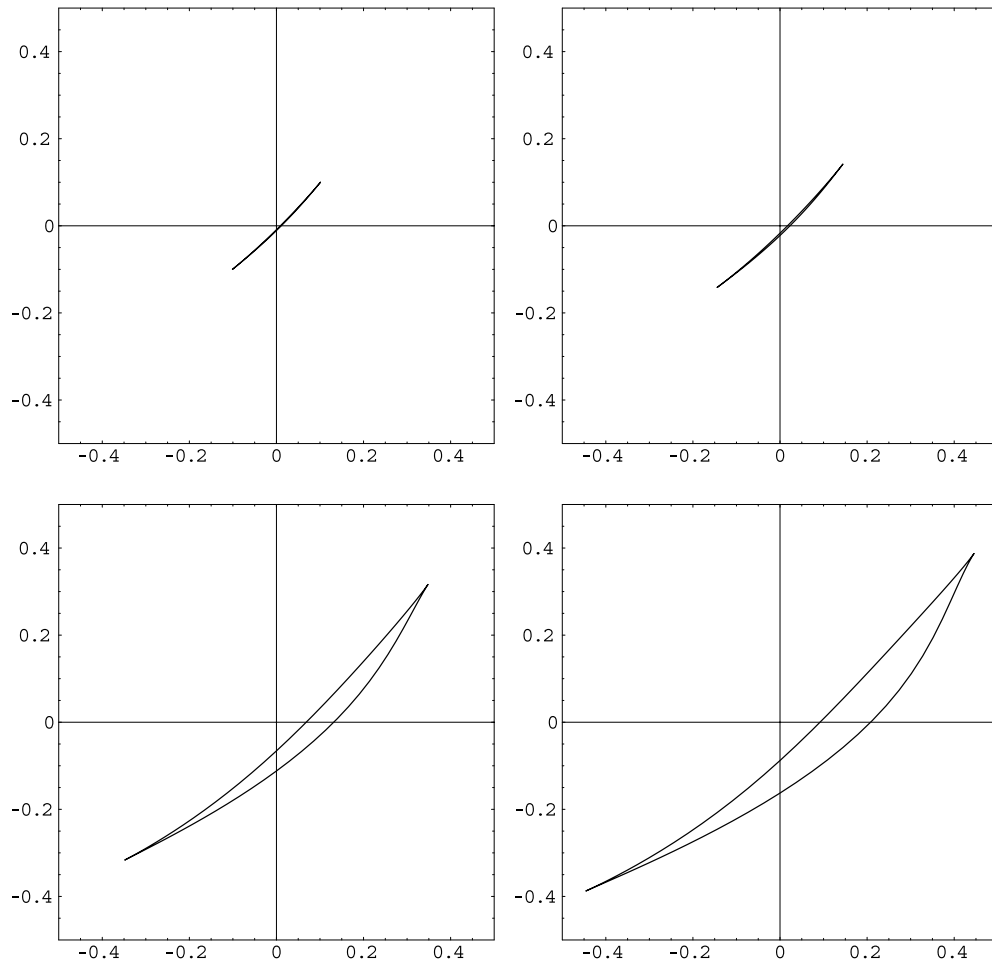


Figure 5. Four consecutive snapshots describing the time evolution of the three-valued region delimited by the caustic, immediately after breaking. The centre of each picture is the breaking point.

coinciding with the $\Delta = 0$ condition. It describes a closed curve of the (\tilde{x}, \tilde{y}) plane possessing two cusps at the points

$$\tilde{x}_c^\pm(\tilde{t}') \sim \tilde{x}_b \mp \sqrt{\frac{|G_\xi| G_{\xi\xi\xi} \epsilon}{\alpha}} \left(\frac{G_{\tilde{y}}}{G_\xi}, 1 \right). \tag{75}$$

If $(x_\pm(\tilde{y}), \tilde{y})$ are the two intersections of this curve with the horizontal line $\tilde{y} = \text{const}$, then

$$\tilde{x}_+(\tilde{y}) - \tilde{x}_-(\tilde{y}) = -\frac{4}{\gamma} |Q|^{3/2} = O(|\epsilon|^{3/2}). \tag{76}$$

It follows that this curve is the boundary of a narrow region of thickness $O(\epsilon^{3/2})$ in the longitudinal direction, and of thickness $O(\epsilon^{1/2})$ in the transversal direction. In figure 5 we show its time evolution after breaking.

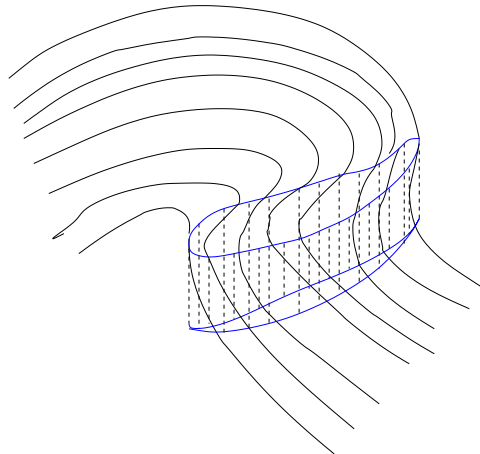


Figure 6. The three-valued solution after breaking, and the caustic delimiting the multivalued region.

Away from the cusps, the solution of the Hopf equation is three-valued, but two of the branches coincide:

$$V_0 = G(\xi_b + \xi'_0, \tilde{y}), \quad V_+ = V_- = G(\xi_b + \xi'_+, \tilde{y}) \tag{77}$$

and the slopes of the coincident solutions are ∞ . We remark that the closed curve, being the envelope of the intersections with the (\tilde{x}, \tilde{y}) -plane of these vertical planes tangent to the wave, is a caustic (see figure 6).

At the two cusps, characterized by the condition $Q = R = 0$, the three real roots of the cubic coincide and the corresponding three solutions of the Hopf equation coincide as well:

$$V_0 = V_+ = V_- = G \left(\xi_b \pm \frac{G_{\xi\xi\tilde{y}}|G_\xi|}{G_{\xi\xi\xi}} \sqrt{\frac{2G_{\xi\xi\xi}t'}{\alpha}}, \tilde{y}_b \pm |G_\xi| \sqrt{\frac{2G_{\xi\xi\xi}t'}{\alpha}} \right). \tag{78}$$

Inside the caustic, the discriminant Δ is strictly negative, the cubic admits three different real roots and the solution of the Hopf equation is three-valued. This is the multivalued region that has to be replaced by a proper shock layer, whose features depend on the desired regularization. Outside the singularity manifold, $\Delta > 0$ and the solution of the Hopf equation is single valued.

We end this section remarking that the similarity solution before breaking, the vertical inflection at breaking, and the caustic after breaking make clear the universal character of the gradient catastrophe of two-dimensional waves evolving according to the Hopf equation. As we shall see in the following sections, similar considerations hold for the gradient catastrophe of two-dimensional waves evolving according to the dKP equation. Similar results on a different model can be found in [31]. A general singularity theory for caustics and wave fronts can be found in [32].

5.2. Longtime breaking of dKP waves

Inverting the transformation (42), the formulae of section 5.1 allow one to describe the longtime breaking of dKP solutions. Now

$$\begin{aligned}
 U(x, y, t) &= \frac{1}{\sqrt{t}} G(\xi, \tilde{y}), \\
 \xi &= x + \frac{y^2}{4t} - 2\sqrt{t}G(\xi, \tilde{y}), \quad \tilde{y} = \frac{y}{2t}
 \end{aligned}
 \tag{79}$$

and

$$\nabla_{(x,y)} U = \frac{1}{\sqrt{t}} \frac{(G_\xi(\xi, \tilde{y}), \frac{y}{2t}G_\xi(\xi, \tilde{y}) + \frac{1}{2t}G_{\tilde{y}}(\xi, \tilde{y}))}{1 + 2\sqrt{t}G(\xi, \tilde{y})}.
 \tag{80}$$

Let $\vec{\xi}_b = (\xi_b, \tilde{y}_b)$ be the breaking parameters and $(\tilde{x}_b, \tilde{y}_b, \tilde{t}_b)$ be the breaking point associated with the solution (44) of the Hopf equation (43), and characterized by equations (47)–(49). Then a localized solution $U(x, y, t)$ of equation (40) first breaks at the time

$$t_b = \left(\frac{\tilde{t}_b}{2}\right)^2 = (2G_\xi)^{-2},
 \tag{81}$$

at the point $\vec{x}_b = (x_b, y_b)$ of the (x, y) -plane given by

$$x_b = \tilde{x}_b - \tilde{y}_b^2 t_b, \quad y_b = 2\tilde{y}_b t_b.
 \tag{82}$$

We remark that \vec{x}_b is the intersection of the parabola $x + y^2/4t_b = \tilde{x}_b$ with the straight line $y = y_b$.

The inflection point $\vec{x}_f(t) = (x_f(t), y_f(t))$ is given by

$$\begin{aligned}
 x_f(t) &= x_b + 2G(\sqrt{t} - \sqrt{t_b}) - \tilde{y}_b^2(t - t_b) \sim x_b + [2|G_\xi|G(1 - \epsilon/4) - \tilde{y}_b^2](t - t_b), \\
 y_f(t) &= y_b + 2\tilde{y}_b(t - t_b),
 \end{aligned}
 \tag{83}$$

where the small parameter ϵ , introduced in (53), reads

$$\epsilon = \frac{t - t_b}{t_b}
 \tag{84}$$

in terms of the dKP coordinates. The inflection point $\vec{x}_f(t)$ is the intersection of the parabola $x + y^2/4t = \tilde{x}_f = \tilde{x}_b + 2G(\sqrt{t} - \sqrt{t_b})$ with the straight line $y = y_f(t)$ (see figure 7), where \tilde{x}_f is defined in (61). At the inflection point, $U_{xx} = U_{xy} = 0$, and U and its gradient take the following values:

$$U = \frac{G}{\sqrt{t}}, \quad \nabla_{(x,y)} U = \sqrt{\frac{t_b}{t}} \frac{1}{t - t_b} \left(1, \tilde{y}_b + \frac{1}{2t} \frac{G_{\tilde{y}}}{G_\xi}\right) \sim \frac{1}{t - t_b} (1, \tilde{y}_b + 2G_{\tilde{y}}G_\xi).
 \tag{85}$$

Throughout this section we make a systematic use of the expressions of the small variables \tilde{x}' , \tilde{y}' , \tilde{t}' in terms of the dKP variables:

$$\begin{aligned}
 \tilde{x}' &= (x - x_b) + \tilde{y}_b \left((y - y_b) - \tilde{y}_b(t - t_b) + \frac{(y - y_b)^2}{2y_b} \right) (1 - \epsilon + O(\epsilon^2)), \\
 \tilde{y}' &= \tilde{y}_b \frac{y - y_b - 2\tilde{y}_b(t - t_b)}{y_b} (1 - \epsilon + O(\epsilon^2)) = \tilde{y}_b \frac{y - y_f(t)}{y_b} (1 - \epsilon + O(\epsilon^2)), \\
 \tilde{t}' &= \frac{t - t_b}{\sqrt{t_b}} (1 - \epsilon/4).
 \end{aligned}
 \tag{86}$$

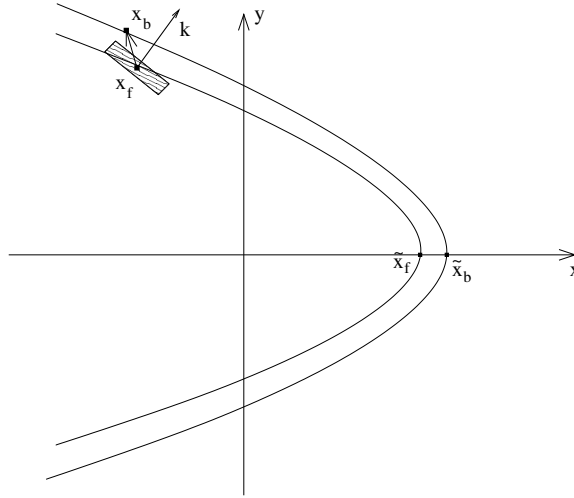


Figure 7. The two parabolas containing the inflection and breaking points, immediately before breaking, together with the narrow strip of the similarity regime.

5.2.1. *Before breaking.* If $t < t_b$, in regime (54), the real solution U of (40) is single valued and described by

$$U(x, y, t) \sim \frac{1}{\sqrt{t}} G(\xi_b + \xi'_0, \tilde{y}_b + \tilde{y}') \tag{87}$$

where ξ'_0 is defined in (55) by Cardano’s formula, and \tilde{x}', \tilde{y}' are given in (86).

To obtain a more explicit solution, we restrict the analysis to the narrow strip around the inflection point, defined by the conditions

$$|X| \sim |\vec{k} \cdot (\vec{x} - \vec{x}_f(t))| = O(|\epsilon|^{p+1}), \quad \frac{|y - y_f(t)|}{t_b} = O(|\epsilon|^q), \tag{88}$$

where p, q are defined in (64) and the vector

$$\vec{k} = (1, \tilde{y}_b + 2G_\xi G_{\tilde{y}}) \tag{89}$$

characterizes the breaking direction. On this strip the cubic linearizes and its solution reads

$$\xi' = \frac{(x - x_b) - (2G|G_\xi| - \tilde{y}_b^2)(t - t_b) + (\tilde{y}_b + 2G_\xi G_{\tilde{y}})(y - y_b - 2\tilde{y}_b(t - t_b))}{\frac{G_\xi}{\sqrt{t_b}}(t - t_b)}, \tag{90}$$

and U is approximated by the exact similarity solution of equation (41):

$$U \sim \frac{x + \frac{y^2}{4t} - \tilde{x}_b + \frac{G_n}{G_\xi}(\frac{y}{2t} - \tilde{y}_b)}{2\sqrt{t}(\sqrt{t} - \sqrt{t_b})}. \tag{91}$$

Equation (91) is equivalent, at the same order, to

$$\begin{aligned} U &\sim \sqrt{\frac{t_b}{t}} \frac{x - x_b + \tilde{y}_b^2(t - t_b) + (\tilde{y}_b + 2G_\xi G_{\tilde{y}})[y - y_b - 2\tilde{y}_b(t - t_b)]}{t - t_b} \\ &\sim \frac{x - x_b + \tilde{y}_b^2(t - t_b) + (\tilde{y}_b + 2G_\xi G_{\tilde{y}})[y - y_b - 2\tilde{y}_b(t - t_b)]}{t - t_b}, \end{aligned} \tag{92}$$

describing the plane tangent to the wave at the point \vec{x}_f (see figure 7). The gradient of U reads, from (80),

$$\nabla_{(x,y)}U = \frac{1}{\sqrt{t}} \frac{(G_\xi, \frac{y}{2t}G_\xi + \frac{1}{2t}G_{\tilde{y}})}{\frac{G_\xi}{\sqrt{t_b}}t' + \frac{G_{\xi\xi\xi}}{2|G_\xi|}[(\xi' + \frac{G_{\xi\xi\tilde{y}}}{G_{\xi\xi\xi}}\tilde{y}')^2 + \frac{\alpha}{G_{\xi\xi\xi}}\tilde{y}'^2]}, \quad (93)$$

where ξ' and \tilde{y}' are given in (90) and (86). On the narrow strip (88) it is very large:

$$\nabla_{(x,y)}U \sim \frac{1}{t - t_b} (1, \tilde{y}_b + 2G_{\tilde{y}}G_\xi), \quad (94)$$

while outside, for $|X| = O(|\epsilon|)$, it is finite.

5.2.2. At breaking. As $t \uparrow t_b$, the inflection point becomes the breaking point: $\vec{x}_f \rightarrow \vec{x}_b$, the above tangent plane becomes vertical, with equation $\vec{k} \cdot (\vec{x} - \vec{x}_b) = 0$ and $U_x, U_y \rightarrow \infty$ as $(t - t_b)^{-1}$.

At the breaking time $t = t_b$, one can give an explicit description of the vertical inflection in the following two subcases.

If $y = y_b$, the solution exhibits the typical vertical inflection preceding the wave breaking:

$$U \sim \frac{1}{\sqrt{t_b}} G(\xi_b + \sqrt[3]{\gamma(x - x_b)}, \tilde{y}_b) \Rightarrow U_x \sim \frac{\sqrt[3]{\gamma}}{3\sqrt{t_b}} \frac{G_\xi}{\sqrt[3]{(x - x_b)^2}}. \quad (95)$$

If $x - x_b + \eta_b(y - y_b) + \frac{(y - y_b)^2}{4t_b} = 0$ (a line approximately tangent to the parabola at \vec{x}_b), U describes a vertical inflection also w.r.t. y :

$$U \sim \frac{1}{\sqrt{t_b}} G\left(\xi_b + \sqrt[3]{2\gamma G_\xi G_{\tilde{y}}(y - y_b)}, \frac{y}{2t_b}\right) \Rightarrow U_y \sim \frac{\sqrt[3]{2\gamma G_\xi G_\eta}}{3\sqrt{t_b}} \frac{G_\xi}{\sqrt[3]{(y - y_b)^2}}. \quad (96)$$

5.2.3. After breaking. If $t > t_b$, in regime (54), the intersection of the SM with any t -constant plane defines an ellipse in the (ξ, η) -plane (see figure 4), corresponding to the following caustic in the (x, y) -plane, defined, as in (74), by

$$\left[3|G_\xi|G_{\xi\xi\xi}^2 X + \frac{G_{\xi\xi\xi}G_{\xi\xi\tilde{y}}G_{\xi\tilde{y}\tilde{y}}}{2} \left(\frac{G_\xi}{G_{\xi\tilde{y}\tilde{y}}} \epsilon + \tilde{y}'^2 \right) \tilde{y}' - \alpha G_{\xi\xi\tilde{y}} \left(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha} \epsilon - \tilde{y}'^2 \right) \tilde{y}' \right]^2 = \alpha^3 \left(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha} \epsilon - \tilde{y}'^2 \right)^3, \quad (97)$$

where now

$$\tilde{y}' = \tilde{y}_b \frac{y - y_f(t)}{y_b},$$

$$\begin{aligned} X = x - x_b - & \left[2G|G_\xi| \left(1 - \frac{1}{4}\epsilon \right) - \tilde{y}_b^2(1 - \epsilon) \right] (t - t_b) + [(\tilde{y}_b + 2G_\xi G_{\tilde{y}})(y - y_f(t)) \\ & + G_\xi^2(y - y_b)^2](1 - \epsilon) + \frac{G_{\tilde{y}\tilde{y}}}{2G_\xi} \tilde{y}_b^2 \left(\frac{y - y_f(t)}{y_b} \right)^2 \\ & - 2|G_\xi|G_{\tilde{y}}\tilde{y}_b(t - t_b) \frac{y - y_f(t)}{y_b} + \frac{G_{\tilde{y}\tilde{y}\tilde{y}}}{6G_\xi} \tilde{y}_b^3 \left(\frac{y - y_f(t)}{y_b} \right)^3. \end{aligned} \quad (98)$$

The caustic exhibits two cusps at the points

$$\vec{x}^\pm(t) \sim \vec{x}_b \mp \sqrt{2 \frac{G_{\xi\xi\xi}}{\alpha} (t - t_b)} \cdot (\tilde{y}_b + 2G_\xi G_\eta, -1). \quad (99)$$

In addition, if $(x^\pm(y), y)$ are the two intersection points of the caustic with the line $y = \text{const}$, we have

$$x^+(y) - x^-(y) = \frac{2\alpha^{3/2}}{3|G_\xi|G_{\xi\xi\xi}} \left(\frac{|G_\xi|G_{\xi\xi\xi}}{\alpha} \epsilon - \left(\frac{y - y_f}{2t_b} \right)^2 \right)^{3/2} = O(|\epsilon|^{3/2}). \tag{100}$$

Therefore the caustic is the boundary of a narrow region of thickness $O(|\epsilon|^{3/2})$ in the longitudinal direction, and of thickness $O(\sqrt{2\frac{G_{\xi\xi\xi}}{\alpha}(t - t_b)})$ in the transversal direction (see figure 5). On it, the discriminant Δ of the cubic is zero and, away from the cusps, the solution of equation (40) is three-valued, two of the branches coincide:

$$U_0 = \frac{1}{\sqrt{t}} G(\xi_b + \xi'_0, \tilde{y}_b + \tilde{y}'), \quad U_+ = U_- = \frac{1}{\sqrt{t}} G(\xi_b + \xi'_+, \tilde{y}_b + \tilde{y}') \tag{101}$$

and the slopes of the coincident solutions are ∞ (see figure 6). At the two cusps, characterized by the condition $Q = R = 0$, the three real roots of the cubic coincide and the corresponding three solutions of equation (40) coincide as well:

$$U_0 = U_+ = U_- = \frac{1}{\sqrt{t}} G \left(\xi_b \pm 2 \frac{G_{\xi\xi\xi} \eta |G_\xi|^{3/2}}{G_{\xi\xi\xi}} \sqrt{\frac{G_{\xi\xi\xi}}{\alpha} (t - t_b)}, \right. \\ \left. \tilde{y}_b \mp 2 |G_\xi|^{3/2} \sqrt{\frac{G_{\xi\xi\xi}}{\alpha} (t - t_b)} \right). \tag{102}$$

Inside the caustic, the discriminant Δ is strictly negative, the cubic admits three different real roots and the solution of equation (40) is three-valued. Outside, $\Delta > 0$ and the solution of equation (40) is single-valued.

The formulae of this section describe, after replacing U by u , the longtime breaking of the dKP solutions u if, for instance, the dKP initial data $u_0(x, y) = u(x, y, 0)$ are small. For small initial data, the inverse spectral transform for dKP, developed in [26], simplifies enormously. The RH spectral data are expressed in terms of the initial data as follows:

$$R_2(\zeta_1, \zeta_2) \sim \frac{1}{\pi i} \int_{\mathbb{R}^2} \frac{d\xi' dy}{\zeta_1 - \xi'} u_{0\xi'}(\xi' + \zeta_2 y, y), \tag{103}$$

and function G , appearing in all formulae of this section, is also given explicitly in terms of u_0 via (26):

$$G(\xi, \eta) \sim \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{d\xi' d\mu dy}{\xi - \mu^2 - \xi'} u_{0\xi'}(\xi' - \eta y, y). \tag{104}$$

Summarizing, we have shown that small and localized initial data evolving according to the dKP equation break in the longtime regime, and we have described the analytic aspects of such breaking in a surprisingly explicit way. The similarity solution before breaking, the vertical inflection at breaking and the caustic after breaking make clear the universal character of such a gradient catastrophe.

6. The gradient catastrophe at finite time

Small and localized initial waves evolving according to the dKP equation break in the longtime regime in the way described in section 5.2. If the initial wave is not small, the gradient catastrophe takes place at a finite time and, in this section, we study the universal features of this phenomenon immediately before breaking.

The analysis is very similar to that of section 5; the main difference is that, while function G , appearing in the implicit equation (44) for the longtime solution of dKP, is an essentially arbitrary smooth function of two variables, function F , appearing in the implicit equation (13) for the solution of dKP at finite time, is a specific function of three variables, connected with the spectral data via (14). As we shall see, this implies that, at breaking, F satisfies an extra condition (equation (121) below) that is instead automatically satisfied when F is replaced by its asymptotic form:

$$F(x, y, t) \rightarrow \frac{1}{\sqrt{t}} G\left(x + \frac{y^2}{4t}, \frac{y}{2t}\right). \quad (105)$$

The inverse problem of the dKP equation, summarized in formulae (13) and (14), defines an implicit system of equations for the solution u :

$$u = F(\zeta, y, t), \quad (106a)$$

$$x = \zeta + 2F(\zeta, y, t)t. \quad (106b)$$

In analogy with the considerations of section 5, the singularity manifold of dKP is the two-dimensional manifold characterized by the equation

$$S(\zeta, y, t) \equiv 1 + 2F_\zeta(\zeta, y, t)t = 0, \quad (107)$$

that can be solved with respect to t , if $F_\zeta + F_{\zeta t}t \neq 0$, giving

$$t = \check{t}(\zeta, y). \quad (108)$$

Since

$$\nabla_{(x,y)} u = \frac{\nabla_{(\zeta,y)} F(\zeta, y, t)}{1 + 2F_\zeta(\zeta, y, t)t}, \quad (109)$$

the gradient of the localized wave becomes infinity on the SM, and the wave ‘breaks’. Let t_b be the first time at which $S = 0$ at a point $\vec{\zeta}_b = (\zeta_b, y_b)$ of the (ζ, y) -plane:

$$1 + 2F_\zeta(\vec{\zeta}_b, t_b)t_b = 0 \quad \Rightarrow \quad t_b = \check{t}(\vec{\zeta}_b); \quad (110)$$

then we obtain the following conditions characterizing the breaking point $(\vec{\zeta}_b, t_b)$:

$$\begin{aligned} 1 + 2t_b F_\zeta(\vec{\zeta}_b, t_b) &= 0 \\ F_\zeta(\vec{\zeta}_b, t_b) &< 0, \quad F_\zeta(\vec{\zeta}_b, t_b) + t_b F_{\zeta t}(\vec{\zeta}_b, t_b) < 0, \\ F_{\zeta\zeta}(\vec{\zeta}_b, t_b) &= F_{\zeta y}(\vec{\zeta}_b, t_b) = 0, \\ F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b) &> 0, \quad \beta \equiv F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b)F_{\zeta yy}(\vec{\zeta}_b, t_b) - F_{\zeta\zeta y}^2(\vec{\zeta}_b, t_b) > 0. \end{aligned} \quad (111)$$

Due to (106b), at the breaking time t_b the wave breaks in the point $\vec{x}_b = (x_b, y_b)$ of the (x, y) -plane defined by

$$x_b = \zeta_b + 2F(\vec{\zeta}_b, t_b)t_b. \quad (112)$$

As before, we evaluate equations (106b) and (107) near breaking, in the regime:

$$x = x_b + x', \quad y = y_b + y', \quad t = t_b + t', \quad \zeta = \zeta_b + \zeta', \quad (113)$$

where x', y', t', ζ' are small, obtaining, at the leading order, the cubic

$$\zeta'^3 + a(y')\zeta'^2 + b(y', t')\zeta' - \gamma X(x', y', t') = 0, \quad (114)$$

where now

$$\begin{aligned}
 a(y') &= \frac{3F_{\zeta\zeta y}}{F_{\zeta\zeta\zeta}} y', & b(y', t') &= 3 \left[\frac{2(F_{\zeta} + t_b F_{\zeta t})}{F_{\zeta\zeta\zeta}} \epsilon + \frac{F_{\zeta yy}}{F_{\zeta\zeta\zeta}} y'^2 \right], \\
 X(x', y', t') &= x' - 2F(\zeta_b, y, t)t' - 2[F(\zeta_b, y, t) - F]t_b \sim x' + \frac{F_y}{F_{\zeta}} y' \\
 &\quad - 2(F + t_b F_t)t' - \frac{F_{yy}}{2|F_{\zeta}|} y'^2 - 2(F_y + t_b F_{yt})y't' - \frac{F_{yyy}}{6|F_{\zeta}|} y'^3, \\
 \gamma &= \frac{6|F_{\zeta}|}{F_{\zeta\zeta\zeta}},
 \end{aligned} \tag{115}$$

and ϵ is given again by (84), corresponding to the maximal balance $|\zeta'|, |y'| = O(|\epsilon|^{1/2})$ and $X = O(|\epsilon|^{3/2})$. In formula (115), and in the rest of this section, all partial derivatives of F whose arguments are not indicated are meant to be evaluated at the breaking point $(\vec{\zeta}_b, t_b)$.

The three roots of this cubic are given by Cardano's formulae (55)–(57), where now

$$Q(y', t') = \frac{2}{F_{\zeta\zeta\zeta}} (F_{\zeta} + t_b F_{\zeta t}) \epsilon + \frac{\beta}{F_{\zeta\zeta\zeta}^2} y'^2. \tag{116}$$

Function \mathcal{S} reads, at the leading order,

$$\mathcal{S} \sim 2(F_{\zeta} + t_b F_{\zeta t})t' + (F_{\zeta\zeta\zeta}\zeta'^2 + 2F_{\zeta\zeta y}y'\zeta' + F_{\zeta yy}y'^2)t_b. \tag{117}$$

For $t < t_b$ ($t' < 0$), since the coefficient Q is strictly positive (see (111) and (116)), so is the discriminant Δ in (57). It follows that ξ'_0 is real and ξ'_{\pm} are complex conjugate roots; therefore the solution of dKP is single-valued and it is described Cardano's formula. In addition, function \mathcal{S} is also strictly positive, implying that the gradient of u is regular near breaking, in the spacetime region (54).

Restricting the asymptotic region in the (x, y) -plane to the narrow strip $|y'| = O(|\epsilon|^q)$, $|X| = O(|\epsilon|^{p+1})$, where p, q are defined by the inequalities (64), then $|\zeta'| = O(|\epsilon|^p)$, ζ'^3 , $a\zeta'^2 \ll b\zeta' \sim \gamma X$, and the expressions for X and ζ' simplify:

$$\begin{aligned}
 X &= x' + \frac{F_y}{F_{\zeta}} y' - 2(F + t_b F_t)t' + O(|\epsilon|^{p+1}), \\
 \zeta' &\sim \frac{x' + \frac{F_y}{F_{\zeta}} y' - 2(F + t_b F_t)t'}{2(F_{\zeta} + t_b F_{\zeta t})t'},
 \end{aligned} \tag{118}$$

implying

$$u \sim F + \frac{F_{\zeta}}{2(F_{\zeta} + t_b F_{\zeta t})} \frac{x' + (F_y/F_{\zeta})y'}{t'}. \tag{119}$$

Since

$$u_{\text{sim}} \equiv \frac{(x - x_b) + (F_y/F_{\zeta})(y - y_b) + c(t - t_b)}{t - t_b} \tag{120}$$

is an exact similarity solution of dKP, where c is an arbitrary constant, comparing (119) and (120), it follows that F must satisfy the following condition at breaking:

$$F_{\zeta} + 2t_b F_{\zeta t} = 0 \quad (\Rightarrow F_{\zeta t} = F_{\zeta}^2), \tag{121}$$

implying that

$$\begin{aligned}
 \mathcal{S} &\sim F_{\zeta}t' + (F_{\zeta\zeta\zeta}\zeta'^2 + 2F_{\zeta\zeta y}y'\zeta' + F_{\zeta yy}y'^2)t_b, \\
 Q(y', t') &= -\frac{2}{F_{\zeta\zeta\zeta}} \epsilon + \frac{\beta}{F_{\zeta\zeta\zeta}^2} y'^2, & \zeta' &\sim \frac{x' + (F_y/F_{\zeta})y' - 2(F + t_b F_t)t'}{F_{\zeta}t'}.
 \end{aligned} \tag{122}$$

Therefore, in the narrow strip

$$\left| x' + \frac{F_y}{F_\zeta} y' - 2(F + t_b F_t) t' \right| = O(|\epsilon|^{p+1}) \quad (123)$$

of the (x, y) -plane, the solution u is described by the exact similarity solution of dKP:

$$u \sim u_{\text{sim}} = \frac{(x - x_b) + (F_y/F_\zeta)(y - y_b) - (F + 2t_b F_t)(t - t_b)}{t - t_b}, \quad (124)$$

while the asymptotic expression for the gradient of u follows from equations (109) and (122):

$$\nabla_{(x,y)} u \sim \frac{\nabla_{(\zeta,y)} F}{F_\zeta t' + \frac{F_{\zeta\zeta\zeta}}{2|F_\zeta|} \left(\frac{x' + (F_y/F_\zeta)y' - 2(F + t_b F_t)t'}{F_\zeta t'} + \frac{F_{\zeta\zeta y}}{F_{\zeta\zeta\zeta}} y' \right)^2 + \frac{\beta}{2|F_\zeta| F_{\zeta\zeta\zeta}} y'^2}. \quad (125)$$

Then

$$\nabla_{(x,y)} u = \frac{1}{t - t_b} (1, F_y/F_\zeta) \quad (126)$$

in the narrow strip (123), while outside, for $|X| = O(|\epsilon|)$, the gradient is finite. The vector $\vec{k} = (1, F_y/F_\zeta)$, orthogonal to the strip, defines the direction of breaking.

In the limit $t \uparrow t_b$, the narrow strip (123) reduces to the breaking point, the above tangent plane becomes vertical, with equation $F_\zeta(x - x_b) + F_y(y - y_b) = 0$, and the slopes $u_x, u_y \rightarrow \infty$ at that point as $(t - t_b)^{-1}$ (see figure 3). As before, we consider two sections in which the description is simpler.

If $y' = 0$, the solution of dKP is described by the typical vertical inflection at the breaking point \vec{x}_b :

$$u \sim F(\zeta_b + \sqrt[3]{\gamma(x - x_b)}, y_b, t_b) \Rightarrow u_x \sim \frac{\sqrt[3]{\gamma}}{3} \frac{F_\zeta}{\sqrt[3]{(x - x_b)^2}}. \quad (127)$$

If $x' = 0$, we have again the vertical inflection at \vec{x}_b :

$$u \sim F\left(\zeta_b - \sqrt[3]{\gamma \frac{F_y}{|F_\zeta|} (y - y_b)}, y_b, t_b\right) \Rightarrow u_y \sim -\frac{1}{3} \sqrt[3]{\gamma \frac{F_y}{|F_\zeta|} \frac{F_\zeta}{\sqrt[3]{(y - y_b)^2}}}. \quad (128)$$

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